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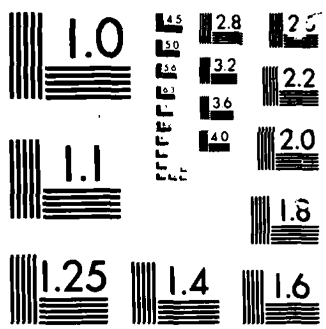

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A CONVERGENCE THEOREM FOR NEWTON-LIKE METHODS IN BANACH SPACES

Tetsuro Yamamoto

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

A convergence theorem for Newton-like methods in Banach spaces is given, which improves results of Rheinboldt [25], Dennis [2], Miel [13, 14] and Moret [16] and includes as a special case an updated version of the Kantorovich theorem for the Newton method given in previous papers [33-35]. Error bounds obtained in [32] are also improved. This paper unifies the study of finding sharp error bounds for Newton-like methods under Kantorovich type assumptions.

AMS (MOS) Subject Classifications: 65G99, 65J15

Key Words: A convergence theorem, error estimates, Newton-like methods, Kantorovich type assumptions

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*Department of Mathematics, Faculty of Science, Ehime University, Matsuyama 790, Japan.

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SIGNIFICANCE AND EXPLANATION

To find sharp error bounds for iterative solution of nonlinear equations in Banach spaces is of basic importance in numerical analysis. This paper gives a convergence theorem for a class of Newton-like methods in Banach spaces, which improves the theorems of Kantorovich [7, 8], Rheinboldt [25], Dennis [2], Miel [13, 14] and Moret [16]. The argument employed in this paper certainly simplifies and unifies the study for finding sharp error bounds for Newton-like methods in Banach spaces.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A CONVERGENCE THEOREM FOR NEWTON-LIKE METHODS IN BANACH SPACES

Tetsuro Yamamoto*

1. Introduction

Let X and Y be Banach spaces and consider an operator $F : D \subseteq X \to Y$ which is Fréchet differentiable in an open convex set $D_0 \subseteq D$. Many iterative methods for solving the equation

$$F(x) = 0 \tag{1.1}$$

can be written in the form

$$x_{n+1} = x_n - A(x_n)^{-1}F(x_n) \quad , \quad n \geq 0 \quad , \tag{1.2}$$

where $x_0 \in D_0$ is given and $A(x)$ denotes a linear operator which approximates the Fréchet derivative $F'(x)$ of F. Under some assumptions, Rheinboldt [25] established a convergence theorem for (1.2) which includes the Kantorovich theorem for the Newton method ($A(x_n) = F'(x_n)$) as a special case. A further generalization was given by Dennis [2, 3]. Miel [13, 14] improved the error bounds for Rheinboldt [25]. Moret [16] obtained a convergence theorem as well as error bounds for the iteration (1.2) under the stronger conditions than those of Rheinboldt. By numerical examples, he showed that his bounds are sharper than those of Miel. *However, no proof* is given. Recently, in [32], we presented a method for finding sharp error bounds for (1.2) under Dennis' assumptions and showed that the bounds obtained improve those of Rheinboldt, Dennis and Miel and reduce to Moret's bounds if we replace the assumptions by his stronger ones. It was also shown that Moret's results can be derived from Rheinboldt's.

In this paper, we first state an updated version of the Kantorovich theorem for the Newton ethod in §2. Next, in §3, we give a simple but powerful principle for finding error bounds for (1.2) under Kantorovich type assumptions. Finally, as an application of this prin^ciple, a convergence theorem for (1.2) is given in §4, which includes the updated

*Department of Mathematics, Faculty of Science, Ehime University, Matsuyama 790, Japan.

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version of the Kantorovich theorem and improves the results of Rheinboldt, Dennis, Miel and Moret. Our approach certainly simplifies and unifies the study for finding sharp error bounds for the Newton-like methods. (Also see Yamamoto [32-35].)

2. An Updated Version of the Kanotorovich Theorem

Let F, D_0 and x_0 be defined as in §1 and $F(x_0) \neq 0$ without loss of generality. Furthermore, assume that $F'(x_0)^{-1}$ exists and the following conditions are satisfied:

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq K\|x-y\|, \; x,y \in D_0, \; K > 0 \quad ,$$

$$\eta = \|F'(x_0)^{-1}F(x_0)\|, \; h = k\eta \leq \frac{1}{2}, \; t^* = \frac{2\eta}{1+\sqrt{1-2h}} \quad ,$$

$$\bar{S} = \bar{S}(x_1, t^*-\eta) = \{x \in X \mid \|x-x_1\| \leq t^*-\eta\} \subseteq D_0 \quad .$$

Under these assumptions, define the scaler sequence $\{t_n\}$ by

$$t_0 = 0, \; t_{n+1} = t_n - f(t_n)/f'(t_n), \; n \geq 0 \quad ,$$

with $f(t) = \frac{1}{2}Kt^2 - t + \eta$, and the sequences $\{B_n\}$, $\{\eta_n\}$ and $\{h_n\}$ by

$$B_0 = 1, \; \eta_0 = \eta, \; h_0 = h = K\eta \quad ,$$

$$B_n = B_{n-1}/(1 - h_{n-1}), \; \eta_n = \frac{1}{2}KB_n\eta_{n-1}^2, \; h_n = KB_n\,\eta_n, \; n \geq 1 \quad .$$

Then, in [34, 35], we obtained the following result, which is an updated version of the Kantorovich theorem and essentially equivalent to [33, Theorem 3.1] with the optimal parameter φ.

Theorem 2.1. With the above notation and assumptions, we have the following:
(i) The Newton method $x_{n+1} = x_n - F'(x_n)^{-1}F(x_n)$ is well defined for every $n \geq 0$, $x_n \in S$(interior of $\bar{S}$) for $n \geq 1$ and $\{x_n\}$ converges to a solution $x^* \in \bar{S}$ of the equation (1.1).
(ii) The solution x^* is unique in

$$\tilde{S} = \begin{cases} S(x_0, t^{**}) \cap D_0 & (2h < 1) \\ \bar{S}(x_0, t^{**}) \cap D_0 & (2h = 1) \end{cases} \quad ,$$

where $t^{**} = (1 + \sqrt{1-2h})/K$ and $S(x_0, t^{**})$ denotes the interior of $\bar{S}(x_0, t^{**})$.

(iii) Let $\bar{S}_0 = \bar{S}$, $\bar{S}_n = \bar{S}(x_n, t^*-t_n)\,(n \geq 1)$,

$$K_n = \sup_{\substack{x,y\in\bar{S}_n \\ x\neq y}} \frac{\|F'(x_n)^{-1}(F'(x) - F'(y))\|}{\|x - y\|} \quad (n \geq 0) \quad ,$$

$$L_n = \sup_{\substack{x,y\in\bar{S} \\ x\neq y}} \frac{\|F'(x_n)^{-1}(F'(x) - F'(y))\|}{\|x - y\|} \quad (n \geq 0) \quad .$$

Then, for $n \geq 0$, we have $t_{n+1} - t_n = \eta_n$ and the following error estimates hold:

$$\|x^* - x_n\| \leq \frac{2d_n}{1 + \sqrt{1-2K_n d_n}}$$

$$\leq \frac{2d_n}{1 + \sqrt{1-2L_n d_n}} \tag{2.1}$$

$$\leq \frac{2d_n}{1 + \sqrt{1-2K(1-K\Delta_n)^{-1} d_n}} \quad \text{(Moret [16])} \tag{2.2}$$

$$\leq \frac{2d_n}{1 + \sqrt{1-2KB_n d_n}} \quad \text{(Kantorovich [7])}$$

$$\leq \frac{2d_n}{1 + \sqrt{1-2h_n}} \quad \text{(Döring [5])}$$

$$= \frac{t^*-t_n}{\nabla t_{n+1}} d_n \quad \text{(Miel [14, 15])}$$

$$\leq t^* - t_n \quad \text{(Kantorovich [8])}$$

$$= \frac{2\eta_n}{1 + \sqrt{1-2h_n}} \quad \text{(Kantorovich [7])}$$

$$\leq \frac{1}{2^{n-1}} (2h)^{2^n-1} \eta \quad \text{(Kantorovich [7])} \quad ,$$

where $d_n = \|x_{n+1} - x_n\|$, $\Delta_n = \|x_n - x_0\|$ and ∇ denotes the backward difference operator.

The well known bounds obtained by Dennis [1], Rall-Tapia [24], Tapia [28], Ostrowski [18, 19], Gragg-Tapia [6] and Potra-Pták [22] fall into the above chart (cf. Yamamoto [31 - 35] and Potra [21]). Furthermore, it was shown in [33 - 35] that (2.1) improves the bounds of Lancaster [11], Kornstaedt [10] and Potra [21]. It is also easy to see that the bound of Potra [20] follows from (2.2). Therefore, Theorem 2.1 gives a unified derivation of the known error bounds for the Newton method under the Kantorovich assumptions.

3. A Principle for Finding Error Bounds for (1.2)

Before extending Theorem 2.1 to the iteration (1.2) we prove the following result.

Theorem 3.1. Let the equation (1.1) have a solution x^* and consider the iteration (1.2). Let x_n and x_{n+1} be defined for some $n \geqq 0$, and a_n, b_n and c_n be nonnegative numbers such that $a_n > 0$, $b_n \leqq 1$ and

$$\|x^* - x_{n+1}\| \leqq \frac{1}{2} a_n \|x^* - x_n\|^2 + b_n \|x^* - x_n\| + c_n \quad .$$

Furthermore, put $d_n = \|x_{n+1} - x_n\|$. If the polynomial

$$\bar{p}(t) = \frac{1}{2} \bar{a}_n t^2 - (1-\bar{b}_n)t + \bar{c}_n + d_n$$

with $\bar{a}_n \geqq a_n$, $1 \geqq \bar{b}_n \geqq b_n$ and $\bar{c}_n \geqq c_n$ has two positive zeroes $\bar{\sigma}_n^*$, $\bar{\sigma}_n^{**}$ such that $\bar{\sigma}_n^* \leqq \bar{\sigma}_n^{**}$, then the polynomial $p(t) = \frac{1}{2} a_n t^2 - (1-b_n)t + c_n + d_n$ also has two positive zeroes σ_n^*, σ_n^{**} such that $\sigma_n^* \leqq \bar{\sigma}_n^* \leqq \bar{\sigma}_n^{**} \leqq \sigma_n^{**}$. If it is known for any reasons that $\|x^* - x_n\| \leqq \bar{\sigma}_n^*$, then we have an improved error estimate $\|x^* - x_n\| \leqq \sigma_n^*$.

Proof. The first assertion of the theorem easily follows from the fact that $\bar{p}(t) \geqq p(t)$ for $t > 0$. To prove the second assertion, we observe that $p(\|x^* - x_n\|) \geqq 0$ since

$$\|x^* - x_n\| - d_n \leqq \|x^* - x_{n+1}\| \leqq \frac{1}{2} a_n \|x^* - x_n\|^2 + b_n \|x^* - x_n\| + c_n \quad .$$

Therefore we have

$$\|x^* - x_n\| \leqq \sigma_n^* \quad \text{or} \quad \|x^* - x_n\| \geqq \sigma_n^{**} \quad .$$

However, if it is known for any reasons that $\|x^* - x_n\| \leqq \bar{\sigma}_n^*$, then the latter case can be excluded: In fact, we have

$$\sigma_n^* < \bar{\sigma}_n^* \leqq \bar{\sigma}_n^{**} < \sigma_n^{**} \quad \text{if} \quad \bar{a}_n + \bar{b}_n + \bar{c}_n > a_n + b_n + c_n$$

and

$$\sigma_n^* = \bar{\sigma}_n^* \leq \bar{\sigma}_n^{**} = \sigma_n^{**} \quad \text{if} \quad \bar{a}_n + \bar{b}_n + \bar{c}_n = a_n + b_n + c_n \quad .$$

Q.E.D.

The usual convergence proof for (1.2) is done with the use of a majorant sequence $\{t_n\}$ due to Rheinboldt [25] such that $\|x_{n+1} - x_n\| \leq t_{n+1} - t_n$ and $t_n \leq t_{n+1} \leq \cdots \to t^*$ as $n \to \infty$. Therefore, by taking $t^* - t_n$ for $\bar{\sigma}_n^*$ in Theorem 3.1, we can apply the theorem to obtain sharper error bounds. The detailed argument will be given in the next section.

4. A Convergence Theorem for (1.2)

Let F, D_0 and x_0 be defined as in §1 and consider the Newton-like method (1.2). According to Dennis [2] and Schmidt [26], we assume the following:

$$\|A(x_0)^{-1}(F'(x) - F'(y))\| \leq K\|x-y\|, \; x,y \in D_0, \; K > 0 \quad ,$$

$$\|A(x_0)^{-1}(A(x) - A(x_0))\| \leq L\|x-x_0\| + \ell, \; x \in D_0, \; L \geq 0, \; \ell \geq 0 \quad ,$$

$$\|A(x_0)^{-1}(F'(x) - A(x))\| \leq M\|x-x_0\| + m, \; x \in D_0, \; M \geq 0, \; m \geq 0 \quad ,$$

$$\ell + m < 1, \; \sigma = \max(1, \frac{L+M}{K}), \; F(x_0) \neq 0 \quad ,$$

$$\eta = \|A(x_0)^{-1}F(x_0)\|, \; h = \sigma K\eta/(1 - \ell - m)^2 \leq 1/2 \quad ,$$

$$t^* = (1 - \ell - m)(1 - \sqrt{1-2h})/(\sigma K) \quad ,$$

$$\tilde{t}^{**} = (1 - m + \sqrt{(1-m)^2 - 2K\eta})/K \quad ,$$

$$\bar{S} = \bar{S}(x_1, t^*-\eta) \subseteq D_0 \quad .$$

Under these assumptions, define the sequence $\{t_n\}$ by

$$t_0 = 0, \; t_{n+1} = t_n + f(t_n)/g(t_n), \; n \geq 0 \quad ,$$

with $f(t) = \frac{1}{2}\sigma K t^2 - (1 - \ell - m)t + \eta$ and $g(t) = 1 - \ell - Lt$, and the sequences $\{p_n\}$, $\{q_n\}$, $\{s_n\}$, $\{\eta_n\}$ and $\{h_n\}$ by

$$p_0 = 1 - \ell,\ q_0 = 1 - \ell - m,\ B_0 = p_0/q_0^2,\ \eta_0 = \eta,\ h_0 = \sigma K B_0 \eta_0 \quad ,$$

$$p_n = 1 - \ell - L \sum_{j=0}^{n-1} \eta_j,\ q_n = 1 - \ell - m - \sigma K \sum_{j=0}^{n-1} \eta_j,\ B_n = p_n/q_n^2 \quad ,$$

$$\eta_n = \{\tfrac{1}{2}\sigma K \eta_{n-1}^2 + (p_{n-1} - q_{n-1})\eta_{n-1}\}/p_n,\ h_n = \sigma K B_n \eta_n,\ n \geq 1 \quad .$$

Furthermore, put

$$\varphi(t) = 1 - \ell - m - (L+M)t,\ \Delta_n = \|x_n - x_0\|$$

and

$$d_n = \|x_{n+1} - x_n\| \quad .$$

Then we can prove the following result, which is a natural generalization of Theorem 2.1.

Theorem 4.1. With the above notation and assumptions, we have the following:

(i) The iteration (1.2) is well defined for every $n \geq 0$, $x_n \in S$(interior of $\bar{S}$) for $n \geq 1$ and $\{x_n\}$ converges to a solution $x^* \in \bar{S}$ of the equation (1.1).

(ii) The solution x^* is unique in

$$\tilde{S} = \begin{cases} S(x_0, \tilde{t}^{**}) \cap D_0 & (\text{if } 2K\eta < (1-m)^2) \\ \bar{S}(x_0, \tilde{t}^{**}) \cap D_0 & (\text{if } 2K\eta = (1-m)^2) \quad . \end{cases} \tag{4.1}$$

(iii) Let $\bar{S}_0 = \bar{S}$, $\bar{S}_n = \bar{S}(x_n, t^* - t_n)(n \geq 1)$,

$$K_n = \sup_{\substack{x,y\in\bar{S}_n \\ x\neq y}} \frac{\|A(x_n)^{-1}(F'(x) - F'(y))\|}{\|x - y\|} (n \geq 0) \quad ,$$

$$L_n = \sup_{\substack{x,y\in\bar{S} \\ x\neq y}} \frac{\|A(x_n)^{-1}(F'(x) - F'(y))\|}{\|x - y\|} (n \geq 0) \quad .$$

Then we have

$$x^* \in \bar{S}_n \subseteq \bar{S}_{n-1} \subseteq \cdots \subseteq \bar{S}_0 \quad ,$$

$$t_{n+1} - t_n = \eta_n,\ 2h_n \leq 1$$

and

$$\|x^* - x_n\| \leq \alpha_n \equiv \frac{2g(\Delta_n)d_n}{\varphi(\Delta_n) + \sqrt{\varphi(\Delta_n)^2 - 2K_n g(\Delta_n)^2 d_n}} \quad (n \geq 0)$$

$$\leq \beta_n \equiv \frac{2g(\Delta_n)d_n}{\varphi(\Delta_n) + \sqrt{\varphi(\Delta_n)^2 - 2L_n g(\Delta_n)^2 d_n}} \quad (n \geq 0)$$

$$\leq \gamma_n \equiv \frac{2g(\Delta_n)d_n}{\varphi(\Delta_n) + \sqrt{\varphi(\Delta_n)^2 - 2Kg(\Delta_n)d_n}} \quad (n \geq 0) \tag{4.2}$$

(Moret [16], Yamamoto [32])

$$\leq \delta_n \equiv \frac{2g(t_n)d_n}{\varphi(t_n) + \sqrt{\varphi(t_n)^2 - 2Kg(t_n)d_n}} \quad (n \geq 0) \tag{4.3}$$

(Yamamoto [32])

$$= \frac{2(p_n/q_n)d_n}{1 + \sqrt{1 - 2KB_n d_n}} \quad (n \geq 0) \tag{4.4}$$

$$\leq \frac{2(p_n/q_n)d_n}{1 + \sqrt{1 - 2\sigma KB_n d_n}} \quad (n \geq 0) \tag{4.5}$$

$$\leq \frac{2(p_n/q_n)d_n}{1 + \sqrt{1 - 2h_n}} \quad (n \geq 0 \tag{4.6}$$

$$= \frac{t^* - t_n}{\nabla t_{n+1}} d_n \quad (n \geq 0) \tag{4.7}$$

$$\leq \frac{t^* - t_n}{\nabla t_n} d_{n-1} \quad (n \geq 1) \qquad \text{(Miel [15])} \tag{4.8}$$

$$= \frac{1}{\eta_{n-1}} \left(\frac{2(p_n/q_n)\eta_n}{1 + \sqrt{1-2h_n}}\right) d_{n-1} \quad (n \geq 1) \tag{4.9}$$

$$\leq t^* - t_n \quad (n \geq 0) \qquad \text{(Rheinboldt [25], Dennis [2])} \tag{4.10}$$

$$= \frac{2(p_n/q_n)\eta_n}{1 + \sqrt{1-2h_n}} \quad (n \geq 0) \quad . \tag{4.11}$$

(iv) Estimates

$$d_{n+1} \leq \frac{1}{g(\Delta_{n+1})} \left\{\frac{1}{2} Kd_n^2 + (m+M\Delta_n)d_n\right\} \leq \frac{\nabla t_{n+2}}{\nabla t_{n+1}} d_n \leq d_n$$

hold, where the last inequality may be replaced by the strict inequality $<$ if $d_n \neq 0$.

Proof. (i) An application of the majorant theory due to Rheinboldt [25] leads to the estimate

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \ n \geq 0 \quad ,$$

from which (i) follows by the standard argument and we obtain (4.10), since $\{t_n\}$ monotonically converges to t^*.

(ii) We have

$$F'(x_0) = A(x_0)\{I + A(x_0)^{-1}(F'(x_0) - A(x_0))\}^{-1}$$

and

$$\|A(x_o)^{-1}(F'(x_0) - A(x_0))\| \leq m < 1$$

by assumption so that

$$\|F'(x_0)^{-1}F(x_0)\| \leq \|F'(x_0)^{-1}A(x_0)\|\cdot\|A(x_0)^{-1}F(x_0)\| \leq \frac{\eta}{1-m} \quad ,$$

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq \|F'(x_0)^{-1}A(x_0)\|\cdot\|A(x_0)^{-1}(F'(x) - F'(y))\|$$

$$\leq \frac{K}{1-m}\|x-y\| \quad , \quad x,y \in D_0 \quad ,$$

and

$$2\left(\frac{K}{1-m}\right)\left(\frac{\eta}{1-m}\right) \leq \frac{2\sigma K\eta}{(1 - \ell - m)^2} \leq 1 \quad .$$

Furthermore, let $\tilde{t}^*$ be the least solution of the equation $\tilde{f}(t) = \frac{1}{2}Kt^2 - (1-m)t + \eta = 0$. Then we have

$$\eta \leq \frac{\eta}{1-m} < \tilde{t}^* \leq t^*$$

since

$$f(t) \geq \tilde{f}(t) \ (t>0), \ \tilde{f}\left(\frac{\eta}{1-m}\right) = \frac{K}{2}\left(\frac{\eta}{1-m}\right)^2 > 0$$

and

$$\frac{\eta}{1-m} < \tilde{t}^{**} \quad \text{(the largest solution of } \tilde{f}(t) = 0) \quad .$$

Therefore we have $t^* - \eta \geq \tilde{t}^* - \frac{\eta}{1-m}$ and

$$\bar{S}(x_1, \tilde{t}^* - \frac{\eta}{1-m}) \subseteq \bar{S}(x_1, t^* - \eta) \subseteq D_0 \quad .$$

Consequently, the assumptions of Theorem 2.1 are satisfied by replacing K, η and t^* in

the theorem by $K/(1-m)$, $\eta/(1-m)$ and $\tilde{t}^*$, respectively. Hence we obtain from Theorem 2.1 (ii) that the solution is unique in the region $\tilde{S}$ defined in (4.1).

(iii) It is easy to see that $x^* \in \overline{S}_n \subseteq \overline{S}_{n-1}$. To obtain the bounds α_n, β_n, γ_n and δ_n, let

$$u_n(t) = \frac{1}{2} K_n t^2 + g(\Delta_n)^{-1}(m + M\Delta_n)t \quad ,$$

$$v_n(t) = \frac{1}{2} L_n t^2 + g(\Delta_n)^{-1}(m + M\Delta_n)t$$

$$w_n(t) = g(\Delta_n)^{-1} \{\frac{1}{2} Kt^2 + (m + M\Delta_n)t\} \quad ,$$

$$y_n(t) = g(t_n)^{-1} \{\frac{1}{2} Kt^2 + (m + Mt_n)t\} \quad ,$$

$$z_n(t) = g(t_n)^{-1} [\frac{1}{2} \sigma Kt^2 + \{m + (\sigma K-L)t_n\}t] + \nabla t_{n+1} - d_n \quad .$$

Then, as is easily seen, we have

$$\|x^* - x_{n+1}\| \leq u_n(\|x^* - x_n\|) \leq v_n(\|x^* - x_n\|) \leq w_n(\|x^* - x_n\|)$$

$$\leq y_n(\|x^* - x_n\|) \leq z_n(\|x^* - x_n\|) \quad .$$

Furthermore, observe that $t^* - t_n$ is the least solution of the equation $Z(t) \equiv z_n(t) - t + d_n = 0$. In fact, we have

$$(1 - \ell - m - \sigma Kt_n)^2 - 2\sigma Kg(t_n)\ \nabla t_{n+1}$$

$$= (1 - \ell - m - \sigma Kt_n)^2 - 2\sigma Kf(t_n)$$

$$= (1 - \ell - m)^2 - 2\sigma K\eta \geq 0 \quad ,$$

so that $Z(t) = 0$ has two positive solutions and

$$g(t_n)Z(t^* - t_n) = \frac{1}{2} \sigma K(t^* - t_n)^2 - (1 - \ell - m - \sigma Kt_n)(t^* - t_n) + f(t_n)$$

$$= 0 \quad . \tag{4.12}$$

This implies $Z(t^*-t_n) = 0$. Similarly we have $Z(t^{**} - t_n) = 0$, where t^{**} is the largest solution of $f(t) = 0$. Since we have already known by (i) that $\|x^* - x_n\| \leq t^* - t_n$, we can apply Theorem 3.1 to obtain

$$\|x^* - x_n\| \leq \alpha_n \leq \beta_n \leq \gamma_n \leq \delta_n \leq t^* - t_n \quad ,$$

where α_n, β_n, γ_n and δ_n denote the largest solutions of the equations

$$U_n(t) = u_n(t) - t + d_n = 0 \quad ,$$

$$V_n(t) = v_n(t) - t + d_n = 0 \quad ,$$

$$W_n(t) = w_n(t) - t + d_n = 0$$

and

$$Y_n(t) = y_n(t) - t + d_n = 0 \quad ,$$

respectively. Next, by induction on n, we shall prove that $t_{n+1} - t_n = \eta_n$. There is nothing to prove for $n = 0$. If $n \geq 1$ and $t_{k+1} - t_k = \eta_k$ is true for every $k \leq n-1$, then we have

$$\begin{aligned}
t_{n+1} - t_n &= g(t_n)^{-1} f(t_n) \\
&= g(t_n)^{-1}[\{f(t_n) - f(t_{n-1}) - f'(t_{n-1})\nabla t_n\} + \{f'(t_{n-1}) + g(t_{n-1})\}\nabla t_n] \\
&= g(t_n)^{-1}[\tfrac{1}{2}\sigma K(\nabla t_n)^2 + \{m + (\sigma K - L)t_{n-1}\}\nabla t_n] \\
&= g\Big(\sum_{j=0}^{n-1} \eta_j\Big)^{-1}\Big[\tfrac{1}{2}\sigma K\eta_{n-1}^2 + \Big\{m + (\sigma K - L)\sum_{j=0}^{n-2} \eta_j\Big\}\eta_{n-1}\Big] \\
&= p_n^{-1}\{\tfrac{1}{2}\sigma K\eta_{n-1}^2 + (p_{n-1} - q_{n-1})\eta_{n-1}\} \\
&= \eta_n \quad ,
\end{aligned}$$

where we understand that $\sum_{j=0}^{-1} \eta_j = 0$. Furthermore, we have

$$\begin{aligned}
h_n &= \sigma K(p_n/q_n^2)\eta_n \\
&= \sigma K[\tfrac{1}{2}\sigma K(\nabla t_n)^2 + \{m + (\sigma K - L)t_{n-1}\}\nabla t_n]/(1 - \ell - m - \sigma K t_n)^2 \\
&= \sigma K[\tfrac{1}{2}\sigma K(\nabla t_n)^2 + \{m + (\sigma K - L)t_{n-1}\}\nabla t_n]/(1 - \ell - m - \sigma K t_n)^2 \\
&= \sigma K[f(t_n) - f(t_{n-1}) - f'(t_{n-1})\nabla t_n + \{f'(t_{n-1}) + g(t_{n-1})\}\nabla t_n]/(1 - \ell - m - \sigma K t_n)^2
\end{aligned}$$

$$= \sigma K f(t_n)/(1 - \ell - m - \sigma K t_n)^2$$

$$= \frac{1}{2}\{(\sigma K t_n)^2 - 2\sigma K(1 - \ell - m)t_n + 2\sigma K\eta\}/(1 - \ell - m - \sigma K t_n)^2$$

$$= \frac{1}{2}\{(1 - \ell - m - \sigma K t_n)^2 + 2\sigma K\eta - (1 - \ell - m)^2\}/(1 - \ell - m - \sigma K t_n)^2$$

$$\leq \frac{1}{2}$$

and, by (4.12) and $z(t^{**} - t_n) = 0$,

$$t^* - t_n = \frac{1 - \ell - m - \sigma K t_n - \sqrt{(1-\ell-m-\sigma K t_n)^2 - 2\sigma K f(t_n)}}{\sigma K}$$

$$= \frac{q_n - \sqrt{q_n^2 - 2\sigma K g(t_n)\nabla t_{n+1}}}{\sigma K}$$

$$= \frac{q_n - \sqrt{q_n^2 - 2\sigma K p_n \eta_n}}{\sigma K}$$

$$= \frac{2(p_n/q_n)\eta_n}{1 + \sqrt{1-2h_n}} .$$

This leads to the estimates

$$\delta_n = \frac{2(p_n/q_n)d_n}{1 + \sqrt{1-2K(p_n/q_n^2)d_n}} \quad (n \geq 0)$$

$$= \frac{2(p_n/q_n)d_n}{1 + \sqrt{1-2KB_n d_n}} \quad (n \geq 0)$$

$$\leq \frac{2(p_n/q_n)d_n}{1 + \sqrt{1 - 2\sigma KB_n d_n}} \quad (n \geq 0)$$

$$\leq \frac{2(p_n/q_n)d_n}{1 + \sqrt{1-2h_n}} \quad (n \geq 0)$$

$$= \frac{2(p_n/q_n)\eta_n}{1 + \sqrt{1-2h_n}} \frac{d_n}{\eta_n} \quad (n \geq 0)$$

$$= (t^* - t_n) \frac{d_n}{\nabla t_{n+1}} \quad (n \geq 0)$$

$$\leq (t^* - t_n) \cdot \frac{d_{n-1}}{\nabla t_n} \ (n \geq 1)$$

$$= \frac{2(p_n/q_n)\eta_n}{1 + \sqrt{1-2h_n}} \cdot \frac{d_{n-1}}{\eta_{n-1}} \ (n \geq 1)$$

$$\leq \frac{2(p_n/q_n)\eta_n}{1 + \sqrt{1-2h_n}} \ (n \geq 0)$$

$$= t^* - t_n \ (n \geq 0)$$

$$\leq 2(p_n/q_n)\eta_n \ (n \geq 0)$$

$$= q_n^{-1}\{\sigma K\eta_{n-1}^2 + 2(p_{n-1} - q_{n-1})\eta_{n-1}\} \ (n \geq 0) \quad ,$$

where we have used Miel's result [14] $d_n/\nabla t_{n+1} \leq d_{n-1}/\nabla t_n$ $(n \geq 1)$.

(iv) The statement (iv) is proved in [32].

Q.E.D.

It is clear that Theorem 4.1 generalizes Theorem 2.1, although the latter was used in the proof of the former. As another result obtained from Theorem 4.1, we have the following corollary, too.

Corollary 4.1.1. Consider the modified Newton method

$$x_{n+1} = x_n - F'(x_0)^{-1}F(x_n) \quad , \quad n \geq 0 \quad , \tag{4.13}$$

where we assume the following:

$$x_0 \in D_0,\ F'(x_0)^{-1} \text{ exists ,}$$

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq K\|x-y\|,\ x,y \in D_0 \ ,$$

$$\eta = \|F'(x_0)^{-1}F(x_0)\| > 0,\ h = K\eta \leq \frac{1}{2} \ ,$$

$$t^* = (1 - \sqrt{1-2h})/K,\ t^{**} = (1 + \sqrt{1-2h})/K \ ,$$

$$\bar{S} = \bar{S}(x_1,\ t^* - \eta) \subseteq D_0 \ .$$

Then:

(i) The iteration (4.13) is well defined fo every $n \geq 0$, $x_n \in S$ for $n \geq 1$ and $\{x_n\}$ converges to a solution of (1.1).

(ii) The solution is unique in

$$\tilde{S} = \begin{cases} S(x_0, t^{**}) \cap D_0 & (2h < 1) \\ \bar{S}(x_0, t^{**}) \cap D_0 & (2h = 1) \end{cases} .$$

(iii) Define the sequence $\{t_n\}$ by

$$t_0 = 0, \; t_{n+1} = \frac{1}{2} Kt_n^2 + n, \; n \geq 0 .$$

Put $\bar{S}_0 = \bar{S}$, $\bar{S}_n = \bar{S}(x_n, t^* - t_n)$ $(n \geq 1)$,

$$K_n = \sup_{\substack{x,y \in \bar{S}_n \\ x \neq y}} \frac{\|F'(x_0)^{-1}(F'(x) - F'(y))\|}{\|x - y\|} \; (n \geq 0) .$$

Then we have

$$\begin{aligned}
\|x^* - x_n\| &\leq \frac{2d_n}{1 - K\Delta_n + \sqrt{(1-K\Delta_n)^2 - 2K_n d_n}} \; (n \geq 0) \\
&\leq \frac{2d_n}{1 - K\Delta_n + \sqrt{(1-K\Delta_n)^2 - 2K_0 d_n}} \; (n \geq 0) \\
&\leq \frac{2d_n}{1 - K\Delta_n + \sqrt{(1-K\Delta_n)^2 - 2Kd_n}} \\
&\leq \frac{2d_n}{1 - Kt_n + \sqrt{(1-Kt_n)^2 - 2Kd_n}} \; (n \geq 0) \\
&\leq \frac{t^* - t_n}{\nabla t_{n+1}} \; (n \geq 0) \\
&\leq \frac{t^* - t_n}{\nabla t_n} d_{n-1} \; (n \geq 1) \\
&\leq t^* - t_n \; (n \geq 0) \\
&= \frac{2(1-Kt_n)^{-1}\nabla t_{n+1}}{1 + \sqrt{1-2K(1-Kt_n)^{-2}\nabla t_{n+1}}} \; (n \geq 0) .
\end{aligned}$$

Finally we remark that the approach employed in this paper is also applicable to other types of iterations.

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A convergence theorem for Newton-like methods in Banach spaces is given, which improves results of Rheinboldt [25], Dennis [2], Miel [13, 14] and Moret [16] and includes as a special case an updated version of the Kantorovich theorem for the Newton method given in previous papers [33-35]. Error bounds obtained in [32] are also improved. This paper unifies the study of finding sharp error bounds for Newton-like methods under Kantorovich type assumptions.

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